The Application of Lagrange Multipliers in Consumer Choice Theory

Využití Lagrangeových multiplikátorů v teorii spotřebitelské volby

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Abstract

This article deals with the consumer choice theory developed by Irving Fisher, Francis Edgeworth, Vilfredo Pareto, and John Hicks. A three-dimensional utility function is presented as an alternative to indifference curves. In mainstream textbooks, the indifference curves together with the budget constraint are used to find the optimum of a consumer graphically at a point where the budget line is a tangent line to an indifference curve. In this article, a vertical cross-section of the three-dimensional utility function and the Lagrange multipliers are applied to find the optimum of a consumer directly from the three-dimensional utility function subject to the budget constraint.

Keywords

consumer choice, consumer optimum, indifference curves, three-dimensional utility function

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Abstrakt

Článek se zabývá teorií spotřebitelské volby vyvinutou Irvingem Fisherem, Francisem Edgeworthem, Vilfredem Paretem a Johnem Hicksem. Trojrozměrnou užitkovou funkci prezentuje jako alternativu k indiferenčním křivkám. V učebnicích hlavního proudu jsou indiferenční křivky spolu s rozpočtovým omezením používány k určení optima spotřebitele, které se nachází v bodě, kde je rozpočtová přímka tečnou indiferenční křivky. V tomto článku je optimum spotřebitele ukázáno přímo pomocí řezu trojrozměrné užitkové funkce podléhající rozpočtovému omezení a pomocí Lagrangeových multiplikátorů.

Klíčová slova

spotřebitelská volba, optimum spotřebitele, indiferenční křivky, trojrozměrná užitková funkce

1 Introduction

Consumer choice theory is one of the most fundamental concepts in economics. After all, it embodies the very definition of economics as a choice among alternatives subject to scarcity. Every year millions of students around the world learn of the indifference curves, the budget line, and the consumer optimum in undergraduate courses of elementary economics. The consumer choice model also serves for the derivation of the shape of the demand curve, one of the most frequently used economics curves. And it is, of course, used to derive the utility-possibility frontier, a key concept in welfare economics. In short, consumer choice theory is a crucial concept in economics.

In this article, the most common definition of economic science is reminded to emphasise the importance of the indifference analysis. Then the history of the discovery of indifference curves in economic thought is described. A three-dimensional utility function is used to demonstrate the fundamentals of consumer choice theory, and Lagrange multipliers are applied as a tool to find the optimal choice of a consumer. A methodological note is taken regarding the usefulness of the application of the utility function in economics vis-à-vis the fact that it is hardly possible to estimate one's utility function. Finally, a recommendation is made regarding the usage of a three-dimensional utility function for didactical and analytical purposes. Briefly, the objective of this paper is to present clearly a possibility of teaching the concept of utility using a three-dimensional utility function subject to a budget constraint.

2 Definition of Economics

Several definitions are used to describe economic science. Probably the most common one is by British economist Lionel Robbins, who wrote that *"Economics is the science which studies human behaviour as a relationship between given ends and scarce means which have alternative uses."*[1]

Similarly, American economist Murray Rothbard later wrote: *"(Human) action involves the employment of scarce means to attain the most valued ends. Man has the choice of using the scarce means for various alternative ends, and the ends that he chooses are the ones he values most highly."*[2] Then he added that *"various ends are ranked in the order of their importance. These scales of preference may be called happiness or welfare or utility or satisfaction or contentment."*[3]

Whatever the definition, it is assumed that there are limited resources and that people make choices about how to spend these resources among alternative uses. This is precisely what consumer choice theory applies in a model where a consumer with a limited budget chooses between two goods: The scarce means is the budget constraint; the alternative uses are the two goods that an individual can buy and consume, and people make choices according to their preferences or to their utility as represented by the indifference curves.

1 *Robbins (1932), p. 15.*

2 *Rothbard (1962), p. 17.*

3 *Rothbard (1962), p. 18.*

3 The history of indifference curves

A typical picture in economics textbooks shows a two-dimensional graph with axes representing quantities of two goods, X and Y, the budget line showing all combinations of the two goods that the consumer can buy within a limited budget and with given prices of the two goods. Indifference curves represent such combinations of the two goods that give the consumer an equal satisfaction. The point at which the budget line is a tangent line to an indifference curve represents the optimal choice because all other combinations lying on the budget line intersect lower indifference curves and therefore provide lower utility.

Before the establishment of the indifference analysis in economic science, economists described utility only as a two-dimensional function where total utility depended on the quantity of only one good consumed. For instance, Alfred Marshall in his Principles of Economics (first published in 1890) formulated the law of diminishing marginal utility: *"The marginal utility of a thing to anyone diminishes with every increase in the amount of it he already has."*[4] Marshall, however, did not think of the utility function as of a function of more independent variables.

The inclusion of alternative goods as independent variables into the utility function was revolutionary because it substantialized the very definition of economics, which is about choices from alternatives.

Consumer choice theory, with indifference curves, a budget line and optimum was developed mainly by Francis Ysidro Edgeworth, Irving Fisher, Vilfredo Pareto, and John Hicks (in this order), whereas Edgeworth was probably the first man who used the term indifference curve.

In 1891 Francis Ysidro Edgeworth published the book *Mathematical Psychics: An Essay on the Application of mathematics to the Moral Sciences*. He defined the system of two consumers X and Y trading two goods in quantities x for y. He formulated their three-dimensional utility functions, with the dimension of utility "sticking up perpendicularly from the paper": *"Let P, the utility of X, one party, = F (x, y), and Π, the utility of Y, the other party, Φ= (x, y)… Consider P – F (x, y) = 0 as a surface, P denoting the length of the ordinate drawn from any point on the plane of xy (say the plane of the paper) to the surface. Consider Π – Φ(x, y) similarly."*[5]

Then he defined the contract curve and coined the term "indifference [curve]" (without actually depicting it): *"It is required to find a point (x, y) such that, in whatever direction we take an infinitely small step, P and Π do not increase together, but that, while one increases, the other decreases. … It is here proposed to call [the locus] the contract curve. … Consider first in what directions X can take a small step …It is evident that X will step only on one side of a certain line, the line of indifference, as it might be called."* (Edgeworth, 1891, p. 21)

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4 *Marshall (1930), p. 93.*

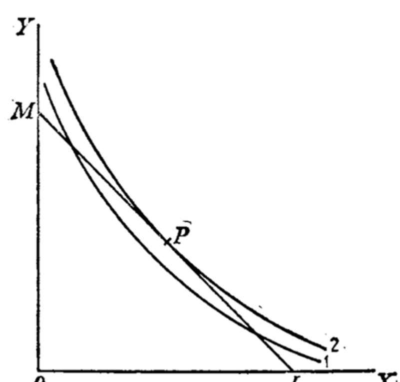
5 *Edgeworth (1891), p. 20-21.*

Irving Fisher used indifference curves in his doctoral dissertation in 1892. In 1926 the work was published under the title *Mathematical Investigations in the Theory of Value and Prices.* Fisher wrote: *"Consider horizontal sections of [three-dimmensional utility function] surface, that is sections parallel to the plane of A and B axes. Each section forms a curve which may be called an indifference curve. It is the locus of points representing all consumption-combinations of A and B which have a given total utility."*[6] and *"[The individual] will select his combination in such manner as to obtain the maximum total utility, which is evidently at the point… where (AB) is tangent to an indifference curve."* [7]

Vilfredo Pareto published his theory of indifference curves in the Italian *Giornale degli Economisti e Annali di Economia* in 1900. He wrote: *"On each point of the plane x, y – supposed horizontal – let us erect perpendiculars equal in length to the [utility] of the point at the foot of the perpendicular. The set of points thus obtained will represent a surface, the indifference lines of which are the projections of the level curves. These curves themselves may be called lines of indifference on the surface."*[8]

John Hicks presented indifference curves in 1930s (Hicks, 1934 and Hicks, 1939). In 1939 John Hicks in his book *Value and Capital* presented a clear graph with axes X and Y representing the quantities of two goods, the budget line and a set of two indifference curves of which one intersects the budget line at two points, and one touches the budget line at the optimum (see Figure 1). Nowadays a similar picture can be found in all mainstream economics textbooks.

Figure 1: Hicks's Consumer optimum with the budget line and the indifference map



Source: Hicks, 1939, p. 16

As Hicks explained, *"It is only when the [budget] line … touches an indifference curve that utility will be maximized. For at a point of tangency, the consumer will get on to a lower indifference curve if he moves in either direction."* [9]

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6 Fisher (1926), p. 70.

7 Fisher (1926), p. 72.

8 Pareto (2008), p. 474.

9 Hicks (1939), p. 17.

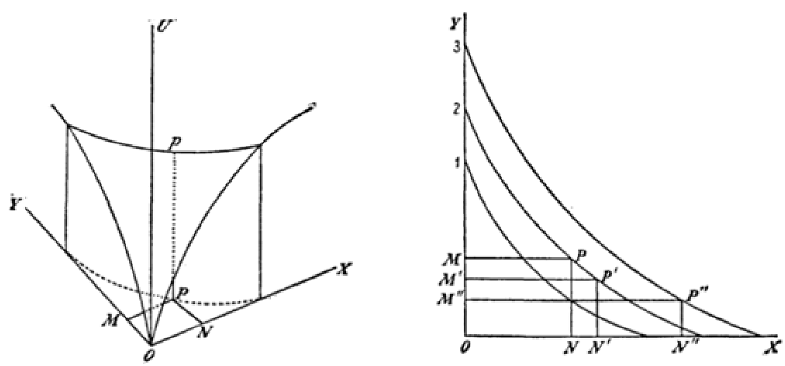
Hicks presented indifference curves as a practical derivation from a three-dimensional utility function. In a three-dimensional graph, the vertical axis represents the total utility and the horizontal axes X and Y represent quantities of the two goods.

Hicks was aware of the fact that drawing and analysing three-dimensional graphs was impractical and complicated. He, therefore, saw a model in which the third dimension was represented by the contour lines – the indifference curves – in a two-dimensional graph as a simplification in comparison with a three-dimensional spatial function. See Figure 2 for Hick's depiction of the utility function both in three dimensions and in two dimensions.

As Hicks put it, *"When we are interested in two commodities, we can draw a utility surface… but three-dimensional diagrams are awkward things to handle. Fortunately having once visited the third dimension we need not stay there. The third dimension can be eliminated, and we can return to two… The contour lines of the utility surface…are the indifference curves."* [10]

Indifference curves are nothing but a conversion of the utility function from three dimensions into more practical two dimensions.

Figure 2: Hicks's Three-dimensional utility function and the indifference curves



Source: Hicks, 1939, p. 15

Since the discovery of indifference curves, economists have not been interested in the three-dimensional utility function anymore, because it is impractical in comparison with the two-dimensional graph of indifference curves.

10 *Hicks (1939), p.13.*

4 A modern display of the three-dimensional utility function

In the 21st century, however, the usage of three dimensions is not that awkward any more. With a wide availability of spreadsheet applications, the utility theory of Edgeworth, Fisher, Pareto, and Hicks can be refreshed, and the three-dimensional utility function can be presented in a modern way.

So, let us formulate the utility function knowing that the indifference map – the set of the infinite number of indifference curves – is a two-dimensional representation of a three--dimensional utility function. Total utility U is the dependent variable, and Q_X and Q_Y, the quantities of the two goods X and Y, are the independent variables.

$$U = f(Q_X, Q_Y)$$

There are some underlying assumptions regarding the utility function.

First, it starts at the origin of the axes since it is assumed that if the consumer consumes none of X and none of Y he gains no satisfaction from consumption.

Second, the function is concave as each additional increase in consumption of any of the goods increases the utility by less than any previous increase.

Third, total utility grows even if the quantity of one of the goods remains at zero. This condition distinguishes the utility function from a technically similar production function where it is necessary, as it is assumed, to employ both factors of production to produce something. Therefore, the Cobb-Douglas function, usually used by economists to describe the production function, does not fit for the description of the utility function.

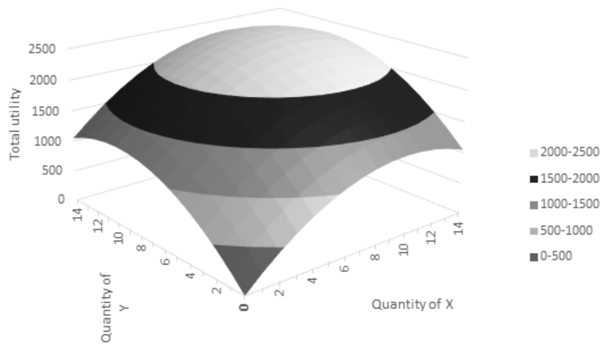
A simple mathematical function complying the above-mentioned assumptions is a quadratic polynomial passing through the origin of the Cartesian coordinate system.

$$U = aQ_X - bQ_X{}^2 + cQ_Y - dQ_Y{}^2$$

This total utility is the sum of the utility gained from the consumption of only X $U_X = aQ_X - bQ_X{}^2$)and the utility gained from the consumption of only Y $(U_Y = cQ_Y - dQ_Y{}^2)$

This utility function can be represented by the three-dimensional graph of a paraboloid (see Figure 3).

Figure 3: A three-dimensional utility function



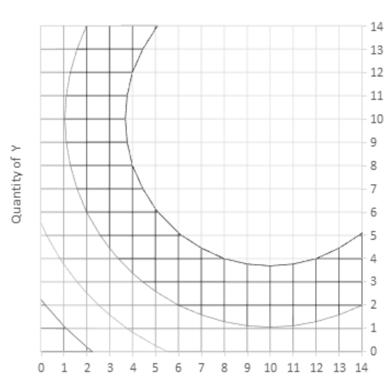
Source: Author

A contour map can be derived out of this three-dimensional graph (see Figure 4), whereas a contour line of a three-dimensional function of two independent variables is a curve along which the function has a constant value.

Like in geography where a three-dimensional landscape can be transformed into a contour map, called a topographic map, a three-dimensional utility function can be transformed into a contour map called an indifference map.

In topography, contour lines represent the points of an equal elevation, while in economics, contour lines called indifference curves join points giving the consumer an equal utility.

Figure 4: Contour lines of a three-dimensional utility function



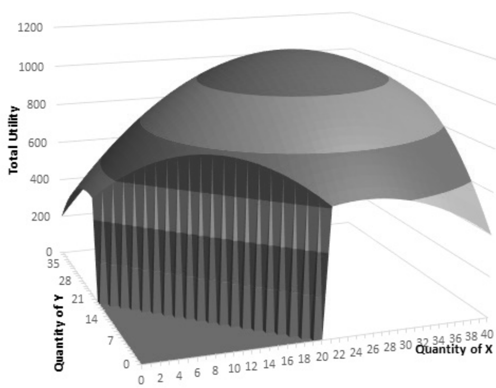
Source: Author

5 A three-dimensional optimum

A graphical method to find the optimum of a three-dimensional function subject to a constraint is to make a vertical cross-section and to find its minimum or maximum.

The consumer optimum (i.e. the maximum of the utility function subject to the budget constraint) is the maximum of the vertical cross-section of the three-dimensional utility function, erected from the budget line (see Figure 5).

Figure 5: The three-dimensional consumer optimum



Source: Author

In a contour graph, the optimum of a consumer lies at the point at which the budget line is a tangent line to an indifference curve, which is the well-known picture of consumer choice in most economics textbooks.

6 Lagrange multipliers

With the reapplication of a three-dimensional utility function, also an algebraic solution to the utility maximisation problem can be applied.

An algebraic method to find the maximum of a multi-dimensional function subject to a constraint is Lagrange multipliers named after Italian-French mathematician Joseph-Louis Lagrange.

This method can be as well applied in economics to find the consumer optimum, subject to a budget constraint.

The optimisation problem here is to find the maximum of the utility function $f(Q_X, Q_Y)$ subject to the budget constraint $g(Q_X, Q_Y)=0$.

A new variable λ (lambda) is introduced to create the Lagrangian expression

$$L = f(Q_X, Q_Y, \lambda) = f(Q_X, Q_Y) - \lambda \cdot g(Q_X, Q_Y)$$

for which the partial derivatives with respect to Q_X and Q_Y ($\partial L/\partial Q_X$ and $\partial L/\partial Q_Y$) are found and set equal to zero.

Together with the budget constraint $g(Q_X, Q_Y) = 0$, three equations with three unknown variables are available. It allows finding the solution – the maximum of the original utility function, subject to the budget constraint.

The budget constraint for two goods is

$$P_X \cdot Q_X + P_Y \cdot Q_Y - \beta = 0$$

where P_X is the price of good X, P_Y is the price of good Y, and β is the size of the budget.

So, we search the solution for the set of the three equations

$$\frac{\partial L}{\partial Q_X} = 0$$

$$\frac{\partial L}{\partial Q_Y} = 0$$

$$P_X \cdot Q_X + P_Y \cdot Q_Y - \beta = 0$$

which gives the optimum Q_X and Q_Y.

Have the following numerical example.

A consumer has a monthly credit of EUR 100 on his mobile phone. He or she spends it on two goods, the text messages and the calls, with a text message costing EUR 0.5 and a call costing EUR 1.0. Assume (arbitrarily) that the utility function describing his satisfaction from texting and calling can be approximated as

$$TU = 180Q_T - Q_T^2 + 160Q_C - Q_C^2$$

The budget constraint is

$$0.5 \cdot Q_T + 1.0 \cdot Q_C - 100 = 0$$

Create the Lagrange function

$$\mathrm{L} = 180Q_T - Q_T^2 + 160Q_C - Q_C^2 - \lambda(0.5 \cdot Q_T + 1.0 \cdot Q_C - 100)$$

Set the partial derivatives equal to zero to find the maximum

$$\frac{\partial \mathrm{L}}{\partial Q_T} = 180 - 2Q_T - 0.5\lambda = 0$$

Hence,

$$Q_T = \frac{180 - 0.5\lambda}{2} = 90 - 0.25\lambda$$

$$\frac{\partial \mathrm{L}}{\partial Q_C} = 160 - 2Q_C - 1.0\lambda = 0$$

Hence,

$$Q_C = \frac{160 - 1.0\lambda}{2} = 80 - 0.5\lambda$$

Substituting this to the budget constraint

$$0.5 \cdot Q_T + 1.0 \cdot Q_C = 100$$

gives

$$0.5 \cdot (90 - 0.25\lambda) + 1.0 \cdot (80 - 0.5\lambda) = 125 - 0.625\lambda = 100$$

$$\lambda = 40$$

Hence,

$$Q_T = 90 - 0.25\lambda = 80$$

$$Q_C = 80 - 0.5\lambda = 60$$

In this example, the consumer will spend his monthly credit of EUR 100 so that he sends 80 text messages and makes 60 calls.

7 Discussion

It can be objected that the usage of a (three-dimensional) utility function is useless and impractical as we can hardly estimate one's utility function.

After all, Pareto believed that bypassing a three-dimensional utility function with two--dimensional indifference curves is practical because, according to Pareto, indifference curves can be found.

It was pointless, according to Pareto, to analyse a three-dimensional utility function which can hardly be measured. It is sufficient to analyse indifference curves themselves, which are "a direct result of experience".

In his *Letters to Maffeo Pantaleoni* in 1899 he gave the following example: *"Here is a child. I ask him: 'Which would you rather have, ten cherries and ten dates, or 9 dates and 11 cherries?' 'I would prefer the first combination.' 'What would you say to 9 cherries and 15 dates?' 'It is the same to me as 10 dates and 10 cherries.' Now I have two points, a and b, of the indifference curves. Others points could be found by the same method."* (Pareto, 1999, p. 171)

We can argue that even if indifference curves, unlike the utility function itself, can be measured directly, they are but a reflection of the utility function. Whether we are able to estimate the utility function or not, the utility function exists.

Consider an analogy with temperature. Before the thermometer was invented, it was difficult or even impossible to measure temperature, but yet different levels of warmth existed. A person does not need to have a thermometer to recognise whether to wear a sweater or just a T-shirt.

Similarly, one does not need to have a "utility-meter" to recognise whether to watch TV one more hour, whether to eat one more cake, or whether to turn up the volume of music by one degree.

People behave as if they knew their utility functions and maximised them subject to a budget or time constraint.

As American economist Milton Friedman put it, a billiard player also does not calculate the strengths and directions of his strikes and still, he is able to aim the ball where he wants it. Although he does not calculate the strikes, the strikes are as if he had calculated them. The same applies to the nature and the behaviour of species, e.g. the position and density of leaves on a tree. The leaves are positioned as if they knew the physical laws and deliberately moved to optimise the received sunlight. We know that neither the billiard player nor the leaves on a tree calculate how to behave, but they behave as if they have calculated it. If they did not, they would have to go and leave a niche to others.

In his book *Essays in Positive Economics,* Friedman wrote: *"Consider the problem of predicting the shots made by an expert billiard player. It seems not at all unreasonable that excellent predictions would be yielded by the hypothesis that the billiard player made his shots as if he knew the complicated mathematical formulas that would give the optimum directions of travel, could estimate accurately by eye the angles, etc., describing the location of the balls, could make lightning calculations from the formulas, and could then make the balls travel in the direction indicated by the formulas."*[11]

Of course, that the billiard player does not calculate directions and strengths, the point is that he behaves as if he did because otherwise, he would not be an expert billiard player.

11 Friedman (1953), p. 12-13.

"Let us turn now to an…example, this time a constructed one designed to be an analogue of many hypotheses in the social sciences. Consider the density of leaves around a tree. I suggest the hypothesis that the leaves are positioned as if each leaf deliberately sought to maximize the amount of sunlight it receives, given the position of its neighbours, as if it knew the physical laws determining the amount of sunlight that would be received in various positions and could move rapidly or instantaneously from any one position to any other desired and unoccupied position."

"Is the hypothesis rendered unacceptable or invalid because, so far as we know, leaves do not "deliberate" or consciously "seek", have not been to school and learned the relevant laws of science or the mathematics required to calculate the "optimum" position, and cannot move from position to position? Clearly, none of these contradictions of the hypothesis is vitally relevant; the phenomena involved are not within the "class of phenomena the hypothesis is designed to explain"; the hypothesis does not assert that leaves do these things but only that their density is the same as if they did." [12]

For the same reason, it entirely makes sense to analyse a three-dimensional utility function in consumer choice theory. Although no one calculates Lagrange multipliers, no one calculates the derivatives, and no one draws down their indifference curves and budget line when shopping, people behave as if they did it.

8 Conclusion and recommendation

This article showed how a three-dimensional display could be useful in economics, specifically in consumer choice theory. It was explained how the mathematical method of the Lagrangian multipliers and the graph of a vertical cross-section could be applied in economics to determine consumer optimum.

The author of this article believes that it would be useful to include a three-dimensional utility function into economics textbooks. With modern graphical methods, displaying three-dimensional functions would help students understand the fundamental principles of utility and consumer choice.

References

EDGEWORTH, F. (1881). *Mathematical Psychics. An Essay on the Application of Mathematics to the Moral Sciences.* London. Available at: https://socialsciences.mcmaster.ca/~econ/ugcm/3ll3/edgeworth/mathpsychics.pdf (Accessed 9 June 2022).

FISHER, I. (1926). *Mathematical Investigations in the Theory of Value and Prices.* New Haven Yale University Press. Available at: https://archive.org/details/in.ernet.dli.2015.84061 (Accessed 9 June 2022).

FRIEDMAN, M. (1953). *Essays in Positive economics*. University of Chicago Press, pp. 3–43. Available at: https://pdfs.semanticscholar.org/4af4/acabcbae145c9d21bca3cfb34fdbb55282a0.pdf (Accessed 9 June 2022).

12 Friedman (1953), p. 12.

HICKS, J. R. (1939). *Value and Capital, an Inquiry into Some Fundamental Principles of Economic Theory.* Oxford University Press. Available at: https://archive.org/details/in.ernet.dli.2015.460445/ (Accessed 9 June 2022).

HICKS, J. R. and R. G. D. ALLEN (1934). A reconsideration of the theory of value. *Economica.* Available at: http://jstor.org./stable/2548574 (Accessed 9 June 2022).

MARSHALL, A. (1930). *Principles of Economics.* Macmillan and Co., London. Available at: https://archive.org/details/in.ernet.dli.2015.217841 (Accessed 9 June 2022).

PARETO, V. (2008). Summary of some chapters of a new treatise on pure economics by Professor Pareto. *Giornale Degli Economisti e Annali Di Economia.* 67 (Anno 121), no. 3, 2008, pp. 453–504. Available at: www.jstor.org/stable/23248332 (Accessed 9 June 2022).

PARETO, V. (1899). Letters to Maffeo Pantaleoni. In: CUNNINGHAM WOOD, J. and M. MCLURE (eds.) (1999): *Vilfredo Pareto. Critical Assessments of Leading Economists.* Routledge.

ROBBINS, L. (1932). *An Essay on the Nature and Significance of Economic Science.* London: Macmillan and Co. Available at: https://mises.org/library/essay-nature-and-significance--economic-science (Accessed 9 June 2022).

ROTHBARD, M. (1962). *Man, Economy, and State.* Ludwig von Mises Institute. Available at: https://archive.org/details/ManEconomyAndStateWithPowerAndMarket2 (Accessed 9 June 2022).

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